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OPTIMAL STRATEGIES FOR SELLING AN ASSET.(U)
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LEVEL IV

OPTIMAL STRATEGIES FOR SELLING AN ASSET

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LEVEL II

OPTIMAL STRATEGIES FOR SELLING AN ASSET

BY

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1. Introduction

This report considers the problem of selling an asset on the open market. As the seller waits for a good offer, he or she receives a random (both in time and in magnitude) sequence of offers. After each offer is received, the seller must decide whether or not to sell, weighing the possibility of obtaining a better offer against the cost of continuing to wait. Successive offers are independent random variables with a common distribution F having finite mean and variance. There is a cost $c > 0$ incurred for each unit of time the asset remains unsold. We will consider three alternative assumptions about the timing of offers. The first is that offers arrive with a fixed, known, periodicity (periodic offer rate). The second is that each period there is a fixed but unknown probability of receiving an offer, which is independent of the size of the offer (geometric offer rate). The third is that the times between offers are independent, identically distributed random variables, with a known distribution (random offer rate). Under the first two assumptions, offers can only be accepted at the start of a period. Under the third assumption, any offer still in force can be accepted at any time. The objective is to maximize the total expected net revenue from the search and sale. We will consider both the case where only the most recently obtained offer may be accepted (no recall), and the case where any previously received offer may be accepted (recall allowed). Also, we will consider both finite and infinite time horizons.

The selling problem is one of several related problems, the most general of which is that of optimally acquiring and divesting assets. This latter problem, however, involves a sequence of ongoing decisions, whereas the selling problem involves only a one-time stopping decision.

The most closely related problem to the selling problem is that of buying an asset. The buying problem, however, is less general in that offers (price quotes) are assumed to arrive periodically. In the selling problem the scenarios of geometric and random offer rates are important, because the seller usually must wait passively for offers to arrive, whereas the buyer may actively search for the best price. The timing of offers is thus an important aspect of this report.

The selling problem arises in many contexts in addition to selling an asset. One is successively interviewing candidates for a job (the secretary problem). Another is in quality assurance, where one sequentially inspects items from a population to find one with an acceptable measure of quality. In this context a periodic offer rate would apply when the inspection process is regular, with a fixed cost or time per inspection. A geometric offer rate would apply when there is a fixed but unknown probability that a given inspection will not yield conclusive results and will have to be repeated. A random offer rate would apply when the inspection process itself is irregular, or when the next time to be inspected must first be located, the locating process taking a random amount of time.

A number of authors have established the existence and properties of optimal search policies when the price distribution, F, is known and the offer rate is periodic:

1) Optimal stopping rules have been shown to exist, both when recall is allowed and under no recall. This result requires only the hypothesis that $\max\{Z,0\}$ have a finite mean and variance, where $Z \sim F$.

2) The optimal stopping rule is characterized by a reservation price, i.e., a price R, possibly dependent upon the number

of periods remaining or originally available, such that the seller should continue to wait if and only if the current best available offer is less than R .

3) The infinite-horizon optimal expected net return exists and is the limit of the finite-horizon optimal expected returns.

4) The optimal stopping rule when recall is allowed is myopic, i.e., the same optimal policy is arrived at if, regardless of the number of periods actually remaining, one acts as if only one period remains. (In this case the reservation price is independent of the time horizon.)

5) When recall is allowed the optimal policy never accepts a previously passed-by offer except possibly in the last period.

6) In the case of no recall, the finite-horizon reservation prices converge to the infinite-horizon reservation price.

This report will investigate conditions under which the above properties hold when the distribution of offers is unknown and the seller's prior distribution of offers undergoes a Bayesian updating as successive offers are received. We will also examine the effects of the alternative assumptions on the timing of offers. For previous work on this and related problems, see Albright [2], DeGroot [4], Derman, Lieberman, and Ross [5], Kohn and Shavell [9], Rothschild [12], and Telser [15]. In general, however, only limited results have been obtained regarding the form of optimal policies.

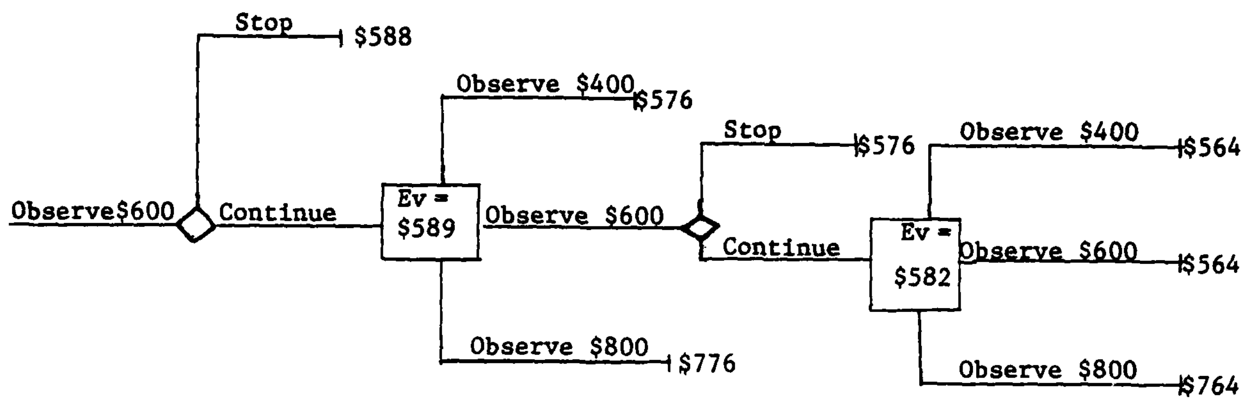
In the recall-allowed case, the main objective of this report is to investigate the efficiency and properties of myopic policies. (Myopic policies have also been investigated by Chow and Robbins [3], Abdel-Hameed [1], and Pratt, Wise, and Zeckhauser [11].) For the no-recall case, the objective is to derive conditions which ensure the optimal policy is reservation (characterized by a reservation price). For the related buying problem, see Rothschild [12], and Rosenfield and Shapiro [13].

In Section 2, we illustrate the greater complexity of optimal policies when the distribution of offers is unknown. In Sections 3 and 4, we consider the cases of recall allowed and no recall, respectively, assuming a periodic offer rate. In Section 5, we investigate extensions to geometric and random offer rates.

2. Some Counterexamples

When the distribution of offers is unknown and the seller updates his or her beliefs about it after each offer is observed, many of the properties established for the known-distribution case may fail to hold unless additional conditions are imposed. This is so even in the simplest case of a periodic offer rate. Consider the following examples.

Example 2.1 - Recall Allowed There exist two possible offer distributions. Each concentrates its mass on two offers, the first on \$400 and \$600, the second on \$600 and \$800. Within each distribution, the lower offer is nine times as likely as the higher offer. The cost per offer is \$12. Suppose that, a priori, the first distribution is nine times as likely to prevail as the second, and that the first offer received is \$600. Then a posteriori, the two price distributions are equally likely. With one period to go, it is optimal to stop and receive a net revenue of \$588, since the alternative of continuing has an expected return of \$586. An analysis of the problem with two periods to go is diagrammed below.



The optimal policy stops if an offer of $800 is received, since it is the maximum offer possible. Also, if, after receiving an offer of $600, an offer of $400 is subsequently observed, the optimal policy stops and recalls the $600 offer, since the $400 offer implies the first price distribution is the one that prevails. With one period to go, the expected value for continuing, having observed $600 twice, is $582. The net revenue from stopping is $576, and so the optimal policy continues. With two periods to go, the expected net revenue from continuing, having observed $600 once, is $589. The net revenue from stopping is $588, and so the optimal policy continues. Since the one-period look-ahead analysis said to stop, the optimal policy is not myopic. Also, the optimal policy is not characterized by a reservation price since it continues after receiving a $600 offer but stops after a $400 offer. Finally, the optimal policy sometimes stops and recalls a previous price (if a $600 offer is followed by a $400 offer) before the final period; this never happens if the price distribution is known.

Example 2.2 - No Recall (due to Rothschild [12]) There exist two possible offer distributions. The first is degenerate at \$3; the second concentrates its mass on \$4 and \$5, with \$5 being far more likely. If an offer of \$3 is received, the optimal policy stops; but if an offer of \$4 is received, the optimal policy continues since the likelihood of soon observing \$5 is high. Thus, no reservation price exists.

3. Unknown Price Distribution with Recall Allowed

We consider in this section and the next the case of a periodic rate of one offer per unit of time. Let $F(\rho|\underline{p})$ be the forecasting distribution of the next offer, given the vector $\underline{p} = (p_1, \ldots, p_N)$ of previously received offers. Let $z(\underline{p}) = \max_{i \leq N}\{p_i\}$ denote the best offer received so far, and $V_t(\underline{p})$ denote the maximum expected net return given a history of offers $\underline{p}$ and a t-period horizon. The recursive relationship which characterizes the finite-horizon maximum expected net return is

$$V_t(\underline{p}) = \max\{z(\underline{p}) , -c + \int V_{t-1} (\underline{p},\rho) \, dF(\rho|\underline{p})\} . \tag{3.1}$$

Following a myopic policy, the stopping condition is obtained from a one-period look ahead:

$$V_1(\underline{p}) = \max\{z(\underline{p}) , -c + \int (z(\underline{p}) \vee \rho) dF(\rho|\underline{p})\} .$$

The stopping condition for this one-period look-ahead policy is

$$z(\underline{p}) > -c + \int (z(\underline{p}) \vee \rho) dF(\rho|\underline{p}) \tag{3.2}$$

or, equivalently

$$c > G(z(\underline{p})|\underline{p}) \ , \tag{3.3}$$

where

$$G(x|\underline{p}) = \int_x^\infty \bar{F}(\rho|\underline{p})d\rho \ ,$$

where

$$\bar{F}(\rho|\underline{p}) = 1 - F(\rho|\underline{p}) \ .$$

If (3.3) determines the optimal stopping rule, then the optimal policy is myopic. A sufficient condition for this to occur is as follows.

Theorem 1 If $G(z(\underline{p})|\underline{p})$ never crosses c from below with additional observations, then the optimal policy (finite and infinite horizons) is myopic.

Proof. The finite-horizon proof is by induction on the number of periods, t. First suppose that (3.3) (or equivalently (3.2)) holds for $\underline{p}$. Then by the hypothesis of the theorem, (3.3) will hold for $(\underline{p}, p_{N+1})$, and so a one-period look-ahead criterion will say stop. Thus, by the induction hypothesis

$$V_{t-1}(\underline{p}, p_{N+1}) = z(\underline{p}) \vee p_{N+1}$$

and with $t-1$ periods to go it is optimal to stop. In this case, the optimal stopping criterion for the t-period problem is

$$-c + \int V_{t-1}(\underline{p},\rho)\ df(\rho|\underline{p}) - z(\underline{p}) = -c + \int (z(\underline{p}) \vee \rho)dF(\rho|\underline{p}) - z(\underline{p}) < 0 \ ,$$

and so stopping is also optimal for the t-period problem. If (3.3) does not hold, then

$$-c + \int V_{t-1}(\underline{p},\rho)\, dF(\rho|\underline{p}) - z(\underline{p}) \geq -c + \int (z(\underline{p}) \vee \rho) dF(\rho|\underline{p}) - z(\underline{p}) \geq 0 ,$$

and it is optimal to continue.

The infinite-horizon optimal net return V is the limit of the finite-horizon returns, so if (3.3) holds

$$V = \lim_{t\to\infty} V_t = z(\underline{p}) \vee p_{N+1} ,$$

and the optimal return is obtained by stopping. As in the finite-horizon proof, when (3.3) does not hold

$$-c + \int V(\underline{p},\rho)\, dF(\rho|\underline{p}) - z(\underline{p}) \geq -c + \int (z(\underline{p}) \vee \rho) dF(\rho|\underline{p}) - z(\underline{p}) \geq 0 ,$$

and it is optimal to continue. $\square$

The quantity $G(z(\underline{p})|\underline{p})$ is the expected gain from one more offer. As Theorem 1 shows, if this quantity never crosses c from below with additional observations, then as soon as the expected gain from one more observation becomes less than the cost to obtain it, the expected gain from <u>any</u> number of additional observations will not cover their cost. Thus a myopic policy prevails. Without this kind of condition on the behavior of $G(z(\underline{p})|\underline{p})$, a one-period look-ahead analysis may be inadequate.

<u>Corollary 1</u>. If $G(z(\underline{p})|p_1,\ldots,p_N)$ depends only upon N and $z(\underline{p})$, and is nonincreasing in each, then the optimal policy is myopic and is characterized by a reservation price. Furthermore, the reservation price is decreasing in N.

<u>Proof</u>. By the monotonicity of $G(z(\underline{p})|\underline{p})$ in N and $z(\underline{p})$, $G(z(\underline{p})|\underline{p})$ decreases with additional observations. Thus by Theorem 1, the optimal policy is myopic. The optimal policy is determined by the sign of

$$G(z(\underline{p})|\underline{p}) - c$$

Let $R_N = \sup\{\rho : G(\rho|p_1,\ldots,p_N) - c \geq 0\}$. Since G is nonincreasing in $z(\underline{p})$ one should stop if and only if $z(\underline{p}) \geq R_N$, and since G is nonincreasing in N, the reservation price R_N is nonincreasing in N.

The monotonicity in N of the reservation price can lead to situations where the reservation price becomes larger than the best possible price and therefore to situations where one should obtain one more offer then stop regardless of what it is. This will be illustrated in an example to follow. First, however, we derive another corollary of Theorem 1.

<u>Corollary 2</u>. Suppose offers are distributed as $F(p+t)$, where t is an unknown translation, and that the posterior distribution for t is a function $h(t|z(\underline{p}),N)$, which depends upon $z(\underline{p})$ and N only. Suppose also that for $z(\underline{p})$ fixed, $h(t|z(\underline{p}),N)$ has monotone likelihood ratio (MLR) in t relative to N†, and for N fixed, $h(u-z(\underline{p})|z(\underline{p}),N)$ has MLR in u relative to $z(\underline{p})$. Then the optimal policy is myopic with nonincreasing reservation prices.

<u>Proof</u>. We establish the result by showing that the conditions of Corollary 1 are satisfied.

$$G(z(\underline{p})|\underline{p}) = \int_{z(\underline{p})}^{\infty} \bar{F}(\rho|\underline{p})d\rho = \int_{z(\underline{p})}^{\infty}\int_{-\infty}^{\infty} \bar{F}(\rho+t)h(t|z(\underline{p}),N)dtd\rho$$

$$= \int_{-\infty}^{\infty} G(z(\underline{p})+t)h(t|z(\underline{p}),N)dt$$

Since $G(z(\underline{p})|\underline{p})$ depends only upon $z(\underline{p})$ and N, all that remains to show is its monotonicity. It can be shown that if a density $f(x|\theta)$ has MLR in x relative to θ and if $\gamma(x)$ is monotonic in x, then

$$\int \gamma(x)f(x|\theta)dx$$

in monotonic in θ in the same direction as γ. Applying this result to the above expression for $G(z(\underline{p})|\underline{p})$ yields that $G(z(\underline{p})|p_1, \ldots p_N)$ is nonincreasing in N. Also,

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† A density $f(x|\theta)$ has monotone likelihood ratio in x relative to θ if and only if for all $\theta_1 > \theta_2$, $f(x|\theta_1)/f(x|\theta_2)$ is nondecreasing in x.

$$G(z(\underset{\sim}{p})|\underset{\sim}{p}) = \int G(u)h(u-z(\underset{\sim}{p})|z(\underset{\sim}{p}), N)du ,$$

so applying the result once more yields that $G(z(\underset{\sim}{p})|\underset{\sim}{p})$ is nonincreasing in $z(\underset{\sim}{p})$. $\square$

<u>Example 3.1 Multinomial Distribution with a Dirichlet Prior</u>

There exist m possible offers $p_1, \ldots, p_m$. The Dirichlet prior is characterized by parameters $N_1, \ldots, N_m$ which are analogous to the frequencies of these offers. The probability of observing offer p_i is N_i/N, where $N = \Sigma N_i$. After observing an offer p_k, the prior distribution is updated by incrementing N_k by one, effectively increasing the probability of observing p_k somewhat and decreasing the probability of the other offers slightly.

Since

$$G(z(\underset{\sim}{p})|p_1, \ldots, p_N) = \sum_{k=z(\underset{\sim}{p})}^{p_{m-1}} (p_{k+1} - p_k) \sum_{j=k+1}^{p_m} N_j/N \tag{3.4}$$

it follows that the multinomial distribution with a Dirichlet prior satisfies the hypothesis of Corollary 1. Thus the optimal policy is myopic and is characterized by a sequence of reservation prices which is nonincreasing in N. To illustrate the one-more-offer-then-stop phenomenon, take $m = 3$, $N_1 = 3$, $N_2 = 2$, $N_3 = 1$, $p_1 = 7$, $p_2 = 14$, $p_3 = 21$, $c = 3.75$. If the first offer is \$7 we continue; but whatever offer is next received we stop, since $G(k|7,k) < c$ for $k = 7, 14$, or 21.

Example 3.2. Exponential Distribution with an Unknown Translation

Let X be a random variable having an exponential distribution $H(x) = 1 - e^{-\lambda(t+x)}$ with an unknown translation t. Let $Z = -2t - X$. This operation re-orients the exponential distribution, putting its mass on $(-\infty, -t]$ instead of $[-t, \infty)$. The distribution of an offer Z is then $F(z) = e^{\lambda(t+z)}$ $(t + z \leq 0)$. Such a distribution may be realistic because often the seller does not know of an explicit upper bound on offers, but may believe that offers are clustered near the unknown upper bound.

Let $r(t)$ be the prior density on the unknown translation, and assume that $r(\cdot)$ is log-concave. Then the posterior density for t can be written as

$$h(t|p_1, \ldots, p_N) = \frac{r(t)\lambda^N \exp(\sum_i (p_i+t))I_{\{z(\underline{p}) + t \leq 0\}}}{\int r(u)\lambda^N \exp(\sum_i (p_i+u))I_{\{z(\underline{p}) + u \leq 0\}}du}$$

$$= K(z(\underline{p}))r(t)e^{\lambda Nt} I_{\{z(\underline{p}) + t \leq 0\}}$$

where $K(z(\underline{p}))$ is a function which depends upon $z(\underline{p})$ only. Thus the posterior density of t can be written as a function of $z(\underline{p})$ and N. For a fixed value of $z(\underline{p})$ and $N \geq M$,

$$\frac{h(t|z(\underline{p}), N)}{h(t|z(\underline{p}), M)} = e^{\lambda(N-M)t}$$

Therefore $h(t|z(\underline{p}), N)$ has MLR in t relative to N. For fixed N and $y \geq x$,

$$\frac{h(u-y \mid y, N)}{h(u-x \mid x, N)} = C(x,y) \frac{r(u-y)}{r(u-x)} ,$$

where $C(x,y)$ depends upon x , y and N , but not u . By the log-concavity of $r(\cdot)$, $r(u-y)/r(u-x)$ is nondecreasing in u , and so $h(u-z(\underline{p}) \mid z(\underline{p}),N)$ has MLR in u relative to $z(\underline{p})$. Thus the conditions of Corollary 2 are satisfied and the optimal policy is myopic with nonincreasing reservation prices.

Example 3.3 Normal Distribution with Unknown Mean

Assume that offers are normally distributed with unknown mean μ and variance 1 , and assume the prior distribution of μ is itself normal with mean μ_0 and variance $1/\tau_0$. Let Φ and ϕ be the distribution and density functions of a standard normal random variable. This example does not satisfy the hypothesis of either corollary. However, as shown in DeGroot [4], if

$$c^2 \geq \frac{\tau_0 + 2}{(\tau_0 + 1)2\pi}$$

then the optimal policy is myopic. Furthermore, the stopping criterion is

$$z(\underline{p}) - \mu(\underline{p}) \geq \frac{\psi^{-1}(c[\tau/(\tau+1)]^{\frac{1}{2}})}{[\tau/(\tau+1)]^{\frac{1}{2}}} ,$$

where $\mu(\underline{p})$ is the posterior mean, $\tau = \tau_0 + N$ is the posterior precision (reciprocal of variance), and

$$\psi(x) = \int_x^\infty [1 - \Phi(z)]dz = \phi(x) - x[1 - \Phi(x)] .$$

The optimal policy for this example may not be reservation. The search may terminate not only upon receiving a very high offer, but also a very low one. (In the latter case the seller would realize offers are much lower than originally thought.)

To summarize, Theorem 1 roughly says that for the optimal policy to be myopic, observing a low offer must cause the mass associated with high offers to decrease. Distributions with origin-related (or, in some cases, mean-related) unknown parameters typically have this property. In general, one would not expect to find myopic optimal policies associated with distributions with unknown spread-related parameters. For example, it can be shown that the optimal stopping rule for a normal price distribution with known mean and unknown variance is not myopic, because observing a very low offer increases the variance estimate and therefore increases the likelihood of subsequently observing a high offer.

4. Unknown Price Distribution with No Recall

The main issue here is when is the optimal policy a reservation-price policy. Rothschild [12] has examined the multinomial distribution with a Dirichlet prior and found the optimal policy to be reservation. Our result is more general in that it is not specific to a particular family of distributions; however, it does not cover the multinomial/Dirichlet.

Definition Given a vector of observed offers, let a sufficient offer be an offer which, if observed next, would cause a seller, following an optimal policy, to stop and sell. Define an insufficient offer similarly.

If the indicated action is strictly preferred over the alternative, we call the offer <u>strictly</u> sufficient or insufficient.

Consider two sellers who have received almost identical sequences of offers (all but one offer identical), and who started with identical priors. Suppose that the seller who received the higher of the non-identical offers has a next-offer distribution which puts more mass on high offers than that of the other seller, and suppose also that the expected gain from one more offer of this seller is not too much greater than that of the other seller. This roughly describes a condition under which a reservation-price policy will prevail.

<u>Theorem 2</u>. Suppose that for all N-component vectors of observed offers $\underset{\sim}{p} \geq \underset{\sim}{q}$ which differ in exactly one component,

$$F(x|\underset{\sim}{p}) \leq F(x|\underset{\sim}{q}) \quad \text{for all} \quad x ,$$

and

$$G(x|\underset{\sim}{q}) \geq G(\underline{x} + \Delta/N|\underset{\sim}{p}) \quad \text{for all} \quad x$$

where Δ denotes the positive component in $\underset{\sim}{p} - \underset{\sim}{q}$. Then

(i) the difference in the sellers' expectations of the next offer observed, Z, is bounded by 0 and Δ/N, i.e.,

$$0 \leq E[Z|\underset{\sim}{p}] - E[Z|\underset{\sim}{q}] \leq \Delta/N . \tag{4.1}$$

(ii) For all t (including $t = \infty$), the differences in the sellers' pre-posterior expectations of the value of continued search is bounded by 0 and Δ/N, i.e.,

$$0 \leq E_{\underline{p}}[V_{t-1}(\underline{p},Z)] - E_{\underline{q}}[V_{t-1}(\underline{q},Z)] \leq \Delta/N . \tag{4.2}$$

In particular, if p_N and q_N are insufficient offers for $p_1, \ldots, p_{N-1}$ and $q_1, \ldots, q_{N-1}$, respectively, then

$$0 \leq V_t(\underline{p}) - V_t(\underline{q}) \leq \Delta/N . \tag{4.3}$$

(iii) The optimal policy is a reservation-price policy.

<u>Proof</u>. For any random variable $E[Y] = \int_0^\infty \overline{F}(y)dy - \int_{-\infty}^0 F(y)dy$. Thus

$$E[Z|\underline{p}] - E[Z|\underline{q}] - \Delta/N = \lim_{T\to-\infty} \int_T^\infty [\overline{F}(\rho|\underline{p}) - \overline{F}(\rho|\underline{q})]d\rho - \Delta/N \tag{4.4}$$

$$\leq \limsup_{T\to-\infty} [\int_T^\infty \overline{F}(\rho|\underline{p})d\rho - \int_{T-\Delta/N}^\infty \overline{F}(\rho|\underline{q})d\rho]$$

$$\leq \limsup_{T\to-\infty} [G(T|\underline{p}) - G(T - \Delta/N|\underline{q})] \leq 0 \quad .$$

This and the stochastic dominance of $F(x|\underline{p})$ over $F(x|\underline{q})$ establishes (i).

The other finite-horizon conclusions of the theorem will be established by induction on the number of periods remaining, using the recursive formula

$$V_t(\underline{p}) = \max\left\{p_N, - c + \int V_{t-1}(\underline{p},\rho)dF(\rho|\underline{p})\right\} . \tag{4.5}$$

For the one-period problem, (ii) follows from (i) since $V_0(\underset{\sim}{p},\rho) = \rho$. To establish (iii), let x be an insufficient offer and y a strictly sufficient offer for $\underset{\sim}{p}$. (If x or y do not exist, the reservation price is trivial.) From the recursion (4.5),

$$E_{(\underset{\sim}{p},y)}[V_0\ (\underset{\sim}{p},y,Z)\ ] < y + c \tag{4.6}$$

$$E_{(\underset{\sim}{p},x)}[V_0\ (\underset{\sim}{p},x,Z)\ ] \geq x + c \tag{4.7}$$

Since $V_0(\underset{\sim}{p},Z) = Z$ for all Z, these inequalities imply

$$E_{(\underset{\sim}{p},y)}[Z] - E_{(\underset{\sim}{p},x)}[Z] < y - x\ .$$

If $y < x$ this contradicts the right-hand inequality of (4.1), which has already been established. Thus $y \geq x$, and the optimal one-period policy is reservation. Now assume (ii) and (iii) hold for the t-period problem, by the induction hypothesis $V_t(\underset{\sim}{q},Z) \leq V_t(\underset{\sim}{p},Z)$. Thus by stochastic dominance

$$E_{\underset{\sim}{q}}[V_t(\underset{\sim}{q},Z)] \leq E_{\underset{\sim}{p}}[V_t(\underset{\sim}{q},Z)] \leq E_{\underset{\sim}{p}}[V_t(\underset{\sim}{p},Z)]\ .$$

This establishes the nonnegativity in (ii). Let P,Q be the t-period reservation prices for $\underset{\sim}{p},\underset{\sim}{q}$, respectively. Then

$$\begin{aligned} &E_{\underset{\sim}{p}}[V_t(\underset{\sim}{p}\,,Z)] - E_{\underset{\sim}{q}}[V_t(\underset{\sim}{q}\,,z)] \\ &\quad = \int_P^\infty \rho f(\rho|\underset{\sim}{p})d\rho - \int_Q^\infty \rho f(\rho|\underset{\sim}{q})d\rho + \int_{-\infty}^P V_t\ (\underset{\sim}{p},\rho)\ f(\rho|\underset{\sim}{p})d\rho \\ &\qquad\qquad - \int_{-\infty}^Q V_t\ (q,\rho)\ f(\rho|\underset{\sim}{q})d\rho\ . \end{aligned}$$

It follows from the induction hypothesis on (ii) that $P \geq Q$. After some algebraic manipulations, the above becomes

$$\int_Q^\infty \rho[f(\rho|p) - f(\rho|q)]d\rho + \int_{-\infty}^Q V_t(q,\rho)\,[f(\rho|p) - f(\rho|q)]d\rho$$
$$+ \int_Q^P [V_t(p,\rho) - \rho]f(\rho|p)d\rho + \int_{-\infty}^Q [V_t(p,\rho) - V_t(q,\rho)\,[f(\rho|p)d\rho\ . \tag{4.8}$$

After integration by parts the first two terms become

$$\int_Q^\infty [\overline{F}(\rho|p) - \overline{F}(\rho|q)]d\rho + \int_{-\infty}^Q \partial/\partial\rho[V_t(q,\rho)\,][\overline{F}(\rho|p) - \overline{F}(\rho|q)]d\rho\ .$$

By the induction hypothesis $\partial/\partial\rho[V_t(q,\rho)\,] \le 1/(N+1)$ for $\rho \le Q$, so using (4.1), the first two terms of (4.8) are bounded above by

$$[1-1(N+1)]\int_Q^\infty [\overline{F}(\rho|p) - \overline{F}(\rho|q)]d\rho + [E[z|p] - E[z|q]]/(N+1)$$
$$\le [G(Q|p) - G(Q|q)]N/(N+1) + \Delta/(N^2+N)\ .$$

Next we establish a bound on the third term of (4.8). For $Q \le \rho \le P$,

$$V_t(p,\rho) - \rho \le V_t(p,Q) - Q + (\rho - Q)[\max_{Q\le\rho\le P}\{\partial/\partial\rho[V_t(p,\rho)\} - 1]$$

Since $V_t(p,Q) - Q = V_t(p,Q) - V_t(q,Q) \le \Delta/(N+1)$

and $\partial/\partial\rho[V_t(p,\rho)] \le 1/(N+1)$ for $Q \le \rho \le P$ by hypothesis,

$$V_t(p,\rho) - \rho \le \Delta/(N+1) - (\rho-Q)N/(N+1) \qquad Q \le \rho \le P \tag{4.9}$$

Evaluating (4.9) for $\rho = P$ and noting that $V_t(p,P) = P$ yields an upper bound on P of $Q + \Delta/N$. Thus

$$\int_Q^P [V_t(p,\rho) - \rho]f(\rho|p)d\rho \le (N+1)^{-1}\int_Q^{Q+\Delta/N}[\Delta - \rho N + QN]f(\rho|p)d\rho\ .$$

Integrating the right-hand side by parts yields an upper bound

$$\bar{F}(Q|\underline{p})\Delta/(N+1) - [G(Q|\underline{p}) - G(Q + \Delta/N|\underline{p})]N/(N+1) .$$

The last term of (4.8) is bounded by $F(Q|\underline{p})\Delta/(N+1)$. Combining these bounds on the terms of (4.8) yields

$$E_{\underline{p}}[V_t(\underline{p},Z)] - E_{\underline{q}}[V_t(\underline{q},Z)] \leq \Delta/N + [G(Q+\Delta/N|\underline{p}) - G(Q|q)]N/(N+1) .$$

The second term of the right-hand side is nonpositive by hypothesis, and so (ii) is established for the $(t + 1)$-period problem. The proof of (iii) for the $(t + 1)$ - period problem follows exactly as in the one-period case by use of (ii).

For $t = \infty$, conclusion (ii) follows as a limiting case of the finite-horizon result. Conclusion (iii) then follows as before. □

If the next-offer distribution given a history of offers $\underline{p}$ is merely a shifted version of that given $\underline{q}$, and if the shift is between 0 and Δ/N in magnitude, then the conditions of Theorem 2 will hold. Thus distributions with unknown location- or mean-related parameters might be expected to have optimal policies which are reservation.

<u>Example 4.1</u> <u>Normal Distribution with Unknown Mean</u>

Let $Z \sim \mathcal{N}(\mu,1)$ with μ an unknown parameter with prior $\mathcal{N}(\mu_0, 1/\tau)$. After observing $\underline{p}$, the posterior on μ is $\mathcal{N}(\mu_N, 1/(\tau+N))$ and the posterior on Z is $\mathcal{N}(\mu_N, 1+1/(\tau+N))$, where

$$\mu_N = (\tau\,\mu_0 + \sum p_i)/(\tau + N) .$$

Since the posterior of Z given $\underset{\sim}{q}$ is $\mathcal{N}(\mu_N - \Delta/(\tau+N)\, ,\, 1 + 1/(\tau + N))$,

$$F(x|\underset{\sim}{q}) = F(x + \Delta/(\tau + N)|\underset{\sim}{p})$$

Thus the hypotheses of Theorem 2 are satisfied.

Example 4.2 Exponential Distribution with an Exponential Prior

Let X have an exponential distribution $F(x|\lambda) = 1 - e^{-\lambda x}$ where λ is an unknown parameter with an exponential prior density $\alpha_0 e^{-\alpha_0 \lambda}$. Let the next offer be $Z = M - X$. (M is the maximum offer possible and is presumed known.) The posterior density of λ given a history of offers $\underset{\sim}{p}$ is

$$h(\lambda|\underset{\sim}{p}) = \frac{(\alpha_N)^{N+1}\lambda^N}{N!} e^{-\alpha_N \lambda}$$

where $\alpha_N = \alpha_0 + \sum(M - p_i)$. The posterior distribution of Z is

$$F(z|\underset{\sim}{p}) = \left(\frac{\alpha_N}{\alpha_N + M - z}\right)^{N+1} .$$

for $\underset{\sim}{p} \geq \underset{\sim}{q}$, $F(z|\underset{\sim}{p}) \leq F(z|\underset{\sim}{q})$, so the first hypothesis of Theorem 2 is satisfied. To verify the second hypothesis, note that

$$G(x|p) = \int_x^M \left[1 - \left(\frac{\alpha_N}{\alpha_N + M - z}\right)^{N+1}\right] dz = \frac{(\alpha_N)^{N+1}}{N(\alpha_N + M - x)^N} + M - \alpha_N/N - x$$

Let k be the index of the nonzero coordinate of $\underset{\sim}{p} - \underset{\sim}{q}$. Then

$$G(x|\underset{\sim}{p}) - G(x - \Delta/N|q) = d(p_k) - d(q_k) ,$$

where

$$d(y) = \frac{(K_1 + M-y)^{N+1}}{N(K_2 + (M-y)(N+1)/N)^N}$$

$$K_1 = \alpha_0 + \sum_{i \neq k} (M-p_i) \ ,$$

and

$$K_2 = K_1 + M - x - (M - p_k)/N \ .$$

Since $d'(y) \leq 0$ for $y \leq p_k$, $d(p_k) - d(q_k) \leq 0$, and so the second hypothesis of the theorem is satisfied.

5. Geometric and Random Offer Rates

The preceding results assumed a periodic offer rate. However, those results involving the multinomial distribution with a Dirichlet prior can be applied to the case of a geometric offer rate by adding a parameter $p_0 = 0$ to describe a null offer and a parameter N_0 to describe the frequency of periods with no offer.

Under certain conditons the preceding results can be applied when the offer rate is random. Let $H(\cdot)$ be the distribution of times between offers under a random offer rate assumption. Let

$$\mu(w) = \frac{\int_w^\infty (\xi - w)\,dF(\xi)}{1 - H(w)}$$

Note that $\mu(0)$ is the mean time between offers, and $\mu(w)$ is the expected remaining time until the next offer given the last offer was received w units of time ago.

Let $V_t(\underline{p})$ be the maximum expected net return given a history of offers $\underline{p}$, a maximal number t of offers which can be considered in the future, and given that an offer has just been received. Supposing an amount of time w has passed since the last offer, the decision on whether or not to accept the best offer still in force is determined by

$$\max\{y(\underline{p}),\ -c\ \mu(w) + \int V_{t-1}(\underline{p},\rho)\,dF(\rho|\underline{p})\}\ ,$$

where

$$y(\underline{p}) = \begin{cases} \max\{p_i\} & \text{recalled allowed} \\ p_N & \text{no recall} \end{cases} .$$

We now show that if $H(\cdot)$ has the following "aging" property, then the only possible times at which an optimal policy will stop are when an offer has just been received.

<u>Definition</u> $H(\cdot)$ <u>is new better than used in expectation</u> (<u>NBUE</u>) if and only if $\mu(w) \leq \mu(0)$ for all $w \geq 0$.

The notion of new better than used in expectation originates in reliability theory. It constitutes one of the weakest notions of aging of physical devices.

<u>Theorem 3</u>. If $H(\cdot)$ is NBUE, then the optimal policy never stops in between offers.

<u>Proof</u>. Given an offer has just been received, the optimal policy continues if and only if

$$y(\underline{p}) < -c\,\mu(0) + \int V_{t-1}(\underline{p},\rho)\,dF(\rho|\underline{p}) .$$

Given w units of time have passed since the last offer, the optimal policy continues if and only if

$$y(\underline{p}) < -c\,\mu(w) + \int V_{t-1}(\underline{p},\rho)\,dF(\rho|\underline{p})$$

since $\mu(0) \geq \mu(w)$ the result follows. $\square$

By Theorem 3, all the results of Sections 3 and 4 can be extended to the case of a random offer rate with NBUE inter-offer times by replacing c by $c\,\mu(0)$.

6. Conclusions

When the offer distribution is unknown, the information obtained from previous offers can influence the distribution of the next offer in a very elaborate fashion. Thus it is not surprising that the optimal policies are generally more complex than in the case of a known distribution. The conditions we have given which guarantee optimal policies which are no more complex than in the known distribution case are clearly restrictive; they would fail to hold for most families of distributions. However, as we have shown, there are important cases where the optimal policies will retain the same simplicity and properties of the known distribution case.

Future research in this area should perhaps consider optimization among a smaller class of policies which are most intuitive and easily implemented, and might seek to develop bounds on the loss resulting from such a policy restriction. The work by Derman, Lieberman, and Ross [5] is a step in this direction. Another area of investigation should be random offer rates when the NBUE condition does not hold.

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This report considers the problem of selling an asset on the open market. The seller receives a random sequence of price offers, which may arrive either periodically or randomly over time. After each offer is received, the seller must decide whether or not to sell, weighing the possibility of obtaining a better offer against the cost of waiting. → next page

(Continued)

cont.

A number of authors have established the properties of optimal selling policies when the distribution of offers is known and the offers are received periodically. This report investigates the conditions under which these same properties hold for an unknown offer distribution which is updated as successive offers are received.

The selling problem has strong similarities but also important differences with the problem of purchasing a commodity subject to an unknown price distribution, and both arise in situations other than buying or selling an asset. Some applications of the model in quality assurance and other settings are briefly discussed.

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